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A Cantor-Bernstein type theorem for effect algebras

Gejza Jenča

ABSTRACT. We prove that if E_1 and E_2 are σ -complete effect algebras such that E_1 is a factor of E_2 and E_2 is a factor of E_1 , then E_1 and E_2 are isomorphic.

1. Introduction

An effect algebra (see [6], [18] and [7]) is a partial algebra $(E; \oplus, 0, 1)$ with a binary partial operation \oplus and two nullary operations 0, 1 satisfying the following conditions.

- (E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
- (E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
- (E3) For every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$.
- (E4) If $a \oplus 1$ exists, then a = 0.

Effect algebras were introduced by Foulis and Bennett in their paper [6]. Independently, Chovanec and Kôpka introduced an essentially equivalent structure called *D*-poset (see [18]). Another equivalent structure was introduced by Giuntini and Greuling in [7].

One can construct examples of effect algebras from any partially ordered abelian group (G, \leq) in the following way: Choose any positive $u \in G$; then, for $0 \leq a, b \leq u$, define $a \oplus b$ iff $a + b \leq u$ and then put $a \oplus b = a + b$. With such a partial operation \oplus , the interval [0, u] becomes an effect algebra $([0, u], \oplus, 0, u)$. Effect algebras which arise from partially ordered abelian groups in this way are called *interval effect algebras*, see [2].

Example 1.1. Let (\mathbb{R}, \leq) be the partially ordered additive group of real numbers, where \leq is the usual partial order. Restrict + to the interval [0,1] in the way indicated in the above paragraph; then $([0,1], \oplus, 0, 1)$ is an effect algebra.

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Another prominent example of an interval effect algebra is the *standard effect* algebra consisting of all bounded self-adjoint operators on a Hilbert space between 0 and the identity operator.

Example 1.2. Let *E* be the four-element set $\{0, a, b, 1\}$. Define $a \oplus a = b \oplus b = 0 \oplus 1$ and $\forall x \in E : 0 \oplus x = x \oplus 0 = x$; in all other cases the partial sum is undefined. Then $(E, \oplus, 0, 1)$ is an effect algebra.

In an effect algebra E, we write $a \leq b$ iff there is $c \in E$ such that $a \oplus c = b$. Since every effect algebra is cancellative, \leq is a partial order on E. In this partial order, 0 is the least and 1 is the greatest element of E. Moreover, it is possible to introduce a new partial operation \ominus ; $b \ominus a$ is defined iff $a \leq b$ and then $a \oplus (b \ominus a) = b$. It can be proved that $a \oplus b$ is defined iff $a \leq b'$ iff $b \leq a'$. Therefore, it is usual to denote the domain of \oplus by \bot .

If E is an effect algebra such that (E, \leq) is a lattice, we say that E is *lattice* ordered. An element a of an effect algebra is called an *atom* iff a is minimal with respect to the property 0 < a.

Example 1.3. The smallest example of a non-lattice ordered effect algebra is a six-element effect algebra with two atoms a, b satisfying $a \oplus b \oplus b = a \oplus a \oplus a = 1$. Note that this equality defines E up to isomorphism.

Among lattice ordered effect algebras, there are two important subclasses, which arise from quantum and fuzzy logic, respectively: orthomodular lattices and MValgebras.

Example 1.4. Let $(L; \land, \lor, \lor, 0, 1)$ be an orthomodular lattice. Write $a \oplus b = a \lor b$ iff $a \leq b'$, otherwise let $a \oplus b$ be undefined. Then $(L, \oplus, 0, 1)$ is an effect algebra. Effect algebras which are associated with orthomodular lattices in this way can be characterized as lattice ordered effect algebras satisfying the implication

$$a \perp b \implies a \wedge b = 0.$$

Example 1.5. An *MV-algebra* (cf. [4], [19]), $(M; \oplus, \neg, 0)$, is a commutative semigroup satisfying the identities $x \oplus 0 = x$, $\neg \neg x = x$, $x \oplus \neg 0 = \neg 0$ and

$$x \oplus \neg (x \oplus \neg y) = y \oplus \neg (y \oplus \neg x).$$

There is a natural partial order in an MV-algebra, given by $y \leq x$ iff $x = x \oplus \neg(x \oplus \neg y)$. For every MV-algebra $(M; \oplus, \neg, 0)$ can be considered as a effect algebra $(M; \oplus, 0, \neg 0)$, when we restrict the operation \oplus to the domain $\bot = \{(x, y) : x \leq \neg y\}$. Effect algebras which are associated with MV-algebras can be characterized as lattice ordered effect algebras satisfying the implication

$$a \wedge b = 0 \implies a \perp b.$$

(Cf. [1])

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As proved by Riečanová in [21], every lattice ordered effect algebra is a union of maximal sub-effect algebras which are MV-algebras. These are called *blocks*. For example, the lattice-ordered effect algebra in Example 1.2 contains two blocks; each of them is a three-element MV-algebra. In the case of of orthomodular lattices, the blocks are maximal Boolean subalgebras. This result shows that the lattice ordered effect algebras are a very natural generalization of orthomodular lattices. In [15], some of the Riečanová's results were generalized for a particular non-lattice ordered class of effect algebras, called *homogeneous* algebras. There is another natural connection with the class of orthomodular lattices. Call an element of an effect algebra sharp iff $a \wedge a' = 0$. The set of all sharp elements in a lattice-ordered effect algebra such that $a \wedge a' = 0$ is an orthomodular lattice. This was proved in [16].

A finite family of elements $A = (a_1, \ldots, a_n)$ of an effect algebra is called *or*thogonal iff $\oplus A = a_1 \oplus \cdots \oplus a_n$ is defined. An infinite family $A = (a_k)_{k \in M}$ is called *orthogonal* iff all finite subfamilies of A are orthogonal. An orthogonal family $A = (a_k)_{k \in M}$ is called *summable* iff

$$\bigoplus A = \bigvee \{a_{i_1} \oplus \cdots \oplus a_{i_n} : \{i_1, \dots, i_n\} \subseteq M\}$$

exists. An effect algebra E is called σ -complete iff every countable orthogonal family of elements of E is summable. Of course, every finite effect algebra is σ -complete. It is easy to see that 1.1 is a σ -complete effect algebra. The interval effect algebra of polynomial functions $[0, 1] \rightarrow [0, 1]$ is not σ -complete. Later we shall prove that a lattice ordered effect algebra E is σ -complete iff (E, \leq) is a σ -complete lattice.

Let E_1, E_2 be effect algebras. Define the \perp and \oplus on $E_1 \times E_2$ as follows: for $a_1, b_1 \in E_1$ and $a_2, b_2 \in E_2$ we have $(a_1, a_2) \perp (b_1, b_2) \Leftrightarrow a_1 \perp b_1$ and $a_2 \perp b_2$, and then $(a_1, a_2) \oplus (b_1, b_2) = (a_1 \oplus b_1, a_2 \oplus b_2)$. Then we say that $(E_1 \times E_2, \oplus, (0, 0), (1, 1))$ is the *direct product* of E_1, E_2 . It is easy to see that the direct product of effect algebras is an effect algebra.

Let us now state the main result of the present paper.

Theorem 1.6. Let E_1, E_2 be σ -complete effect algebras. Suppose that there exist effect algebras F, G such that

$$E_1 \cong E_2 \times F$$
$$E_2 \cong E_1 \times G$$

Then $E_1 \cong E_2$.

The assumption that E_1 and E_2 are σ -complete cannot be dropped, since (as proved by Hanf in [12]) there is a Boolean algebra A such that $A \times A \times A \cong A$, but $A \times A \cong A$. Then we may put $E_1 = A$ and $E_2 = A \times A$ to obtain a counterexample.

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Recently, two more special versions of the main Theorem of [13] appeared. In [23], it was proved that A_1 and A_2 can be σ -complete orthomodular lattices. In [22], Theorem 4.1, it was proved that A_1 and A_2 can be σ -complete MV-algebras.

On the International Conference on Fuzzy Set Theory an its Application (Liptovký Ján 2000), A. De Simone, D. Mundici, P. Pták and M. Navara raised the question, whether Theorem 1.6 holds in a subclass of effect algebras.

Section 2 contains basic definitions and relationships concerning effect algebras and some more general structures. We characterize the class of partial abelian monoids which are partially ordered in their algebraic preorder. In section 3, we focus on the class of σ -complete effect algebras and weak congruences on them. We prove that, for a σ -complete effect algebra E and a countably additive weak congruence \sim satisfying a certain condition, the algebraic preorder of the quotient partial abelian monoid E/\sim is a partial order. In section 4, we prove that the center (see [9]) of a σ -complete effect algebra is a σ -complete Boolean algebra. We apply a result from section 2 to prove that the center of a σ -complete effect algebra modulo the isomorphism of central ideals forms a partial abelian monoid which is partially ordered. Theorem 1.6 is then a simple consequence of the latter result.

2. Effect algebras and other partial abelian monoids

An effect algebra need not be lattice ordered. However, as proved in [9], the following relationship between \land , \lor and \oplus holds: if $a \lor b$ exists and $a \perp b$, then $a \land b$ exists and

$$a \oplus b = (a \land b) \oplus (a \lor b).$$

Moreover, it is easy to check that, for every subset B of an effect algebra such that $\bigvee B$ exists and for every $x \ge B$,

$$x \ominus (\bigvee B) = \bigwedge \{ x \ominus b : b \in B \}.$$

Let E_1, E_2 be effect algebras. A map $\phi: E_1 \mapsto E_2$ is called a homomorphism iff it satisfies the following condition.

(H1) $\phi(1) = 1$ and if $a \perp b$, then $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

A homomorphism $\phi: E_1 \mapsto E_2$ of effect algebras is called *full* iff the following condition is satisfied.

(H2) If $\phi(a) \perp \phi(b)$, then there exist $a_1, b_1 \in E_1$ such that $a_1 \perp b_1, \phi(a) = \phi(a_1)$ and $\phi(b) = \phi(b_1)$.

A bijective, full homomorphism is called an *isomorphism*.

Let E_1 be an effect algebra. A subset $E_2 \subseteq E_1$ is a subeffect algebra of E_1 iff $0, 1 \in E_2, E_2$ is closed under the ' operation, and $a, b \in E_2$ with $a \perp b \implies a \oplus b \in E_2$.

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Another possibility of creating a substructure of an effect algebra E is to restrict \oplus to an interval

$$[0, a] = \{ x \in E : 0 \le x \le a \},\$$

where $a \in E$, so that $([0, a], \oplus)$ becomes a *relative subalgebra* (cf. e.g. [8]) of the partial algebra (E, \oplus) . We can then consider [0, a] as an effect algebra, letting a act as the unit element. In what follows, we denote such effect algebras by $[0, a]_E$.

Let E be an effect algebra. A subset I of E is called an ideal of E iff the following condition is satisfied.

$$x, y \in I$$
 and $x \perp y \Leftrightarrow x \oplus y \in I$

A partial abelian monoid is a partial algebra $(P, \oplus, 0)$ satisfying conditions (E1) and (E2), with a neutral element 0. A partial abelian monoid P is said to be positive iff P satisfies the following condition.

If
$$a \oplus b = 0$$
, then $a = 0$

It is possible to define \leq for a general partial abelian monoid exactly the same way as for effect algebras. However, the relation \leq need not be a partial order.

Proposition 1. Let P be a partial abelian monoid. Then \leq is a partial order on P iff for all $a, b, c \in P$ the following condition is satisfied.

$$a = a \oplus b \oplus c \implies a = a \oplus b$$

Proof. For every partial abelian monoid, \leq is a preorder, i.e., a reflexive and transitive relation. Thus, \leq is a partial order iff \leq is antisymmetric.

Assume that \leq is a partial order and let $a = a \oplus b \oplus c$. Since $a \leq a \oplus b$ and $a \oplus b \leq a, a = a \oplus b$.

For the 'if' part, let $a \leq d$ and $d \leq a$. There are $b, c \in P$ such that $a \oplus b = d$ and $d \oplus c = a$. Thus, $a \oplus b \oplus c = a$ and, by assumption, this implies that $a \oplus b = a$. Therefore, d = a and we see that \leq is a partial order on P.

Corollary 2.1. Let P be a partial abelian monoid such that \leq is a partial order on P. Then P is positive.

Proof. Let $x, y \in P$, $x \oplus y = 0$. Then $x \oplus y \oplus x = x$. By Proposition 1, this implies that $x \oplus y = x$. Thus, by assumption, x = 0.

Obviously, every cancellative and positive partial abelian monoid satisfies the conditions of Proposition 1.

3. Weak congruences and σ -complete effect algebras

Let E be an effect algebra. A relation \sim on E is a *weak congruence* iff the following conditions are satisfied.

(C1) \sim is an equivalence relation.

(C2) If $a_1 \sim a_2$, $b_1 \sim b_2$, $a_1 \perp b_1$, $a_2 \perp b_2$, then $a_1 \oplus b_1 \sim a_2 \oplus b_2$.

If E is an effect algebra and \sim is a weak congruence on E, the quotient E/\sim (\oplus is defined on E/\sim in an obvious way) need not to be a partial abelian monoid, since the associativity condition may fail (cf. [11]). This fact motivates the study of sufficient conditions for a weak congruence to preserve associativity. The following condition was considered in [5].

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(C5) If $b \perp c$ and $a \sim b \oplus c$, then there are b_1, c_1 such that $b_1 \sim b, c_1 \sim c, b_1 \perp c_1$ and $a = b_1 \oplus c_1$.

In [5], it was proved that for a partial abelian monoid P and a weak congruence \sim , satisfying (C5), the quotient P/\sim is again a partial abelian monoid. Moreover, it is easy to prove that the eventual positivity of P is preserved for such \sim . However, for an effect algebra E, the (C5) property of \sim does not guarantee that the ' operation is preserved by \sim or, equivalently, that E/\sim is an effect algebra. We refer the interested reader to [20] and [11] for further details concerning congruences on effect algebras and partial abelian monoids.

Example 3.1. Let $2^{\mathbb{N}}$ be the effect algebra associated with the Boolean algebra of all subsets of \mathbb{N} . If a, b are two subsets of \mathbb{N} , put $a \sim b$ iff a and b are of the same cardinality. Then \sim is a weak congruence satisfying (C5) and $2^{\mathbb{N}}/\sim$ is isomorphic to the partial abelian monoid $(\mathbb{N} \cup \{\infty\}; +, 0)$.

Example 3.2. Let A be an involutive ring with unit, in which $x^*x + y^*y = 0$ implies x = y = 0. Let P(A) be the set of all projections in A. For $e, f \in P(A)$, write $e \oplus f = e + f$ iff ef = 0, otherwise let $e \oplus f$ be undefined. Then $(P(A); \oplus, 0, 1)$ is an effect algebra. For e, f in P(A), write $e \sim f$ iff there is $w \in A$ such that $e = w^*w$ and $f = ww^*$. Then \sim is a weak congruence on P(A) and \sim satisfies (C5).

Proposition 2. Let E be an effect algebra. Let \sim be a weak congruence satisfying (C5). Then \leq is a partial order on E/\sim iff for all $a, b, c \in E$ the following condition is satisfied.

$$a \sim a \oplus b \oplus c \implies a \sim a \oplus b$$
 (1)

Proof. By Proposition 1, \leq is a partial order on E/\sim iff

$$[a]_{\sim} = [a]_{\sim} \oplus [b]_{\sim} \oplus [c]_{\sim} \implies [a]_{\sim} = [a]_{\sim} \oplus [b]_{\sim}$$

$$\tag{2}$$

is satisfied. Thus, it suffices to prove that (1) and (2) are equivalent.

Assume that (2) is satisfied. Let $a, b, c \in P$ be such that $a \sim a \oplus b \oplus c$. Obviously, $[a]_{\sim} = [a]_{\sim} \oplus [b]_{\sim} \oplus [c]_{\sim}$. By (2), this implies $[a]_{\sim} = [a]_{\sim} \oplus [b]_{\sim}$. By assumption, $a \perp b$ so we can write $a \sim a \oplus b$.

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On the other hand, assume that (1) is satisfied. Let $a, b, c \in P$ be such that $[a]_{\sim} = [a]_{\sim} \oplus [b]_{\sim} \oplus [c]_{\sim}$. Since $[a]_{\sim} \perp [b]_{\sim}$, there are $a_1 \sim a$ and $b_1 \sim b$ such that $a_1 \perp b_1$. Since $[a_1 \oplus b_1]_{\sim} \perp [c]_{\sim}$, there are $c_1 \sim c$ and $d \sim a_1 \oplus b_1$, such that $c_1 \perp d$. Since \sim satisfies (C5), $d \sim a_1 \oplus b_1$ implies that there are $a_2, b_2 \in P$ such that $a_2 \sim a_1, b_2 \sim b_1$ and $d = a_2 \oplus b_2$. Thus, $a_2 \oplus b_2 \oplus c_1 \sim a_2$. By (1), this implies that $a_2 \oplus b_2 \sim a_2$. Since $[a]_{\sim} = [a_2]_{\sim}$ and $[b]_{\sim} = [b_2]_{\sim}$, we see that $[a]_{\sim} = [a]_{\sim} \oplus [b]_{\sim}$.

Let E be a σ -complete effect algebra. Let $(a_k)_{k \in \mathbb{N}}$ be an orthogonal family. It is easy to check that,

$$\bigoplus_{k\in\mathbb{N}}a_k=\bigvee_{k\in\mathbb{N}}a_1\oplus\cdots\oplus a_k.$$

The following lemma gives a useful characterization of σ -complete effect algebras. Although it was proved in [10], we include the proof here because we need to refer to it in what follows.

Lemma 3.3. [10] Let E be an effect algebra. The following are equivalent:

(a) E is σ -complete.

(b) For each non-increasing sequence $(a_k)_{k \in \mathbb{N}}$, $\wedge (a_k)_{k \in \mathbb{N}}$ exists.

(c) For each non-decreasing sequence $(a_k)_{k\in\mathbb{N}}$, $\forall (a_k)_{k\in\mathbb{N}}$ exists. *Proof.*

 $(c) \Rightarrow (a)$: Let $(b_k)_{k \in \mathbb{N}}$ be an orthogonal family. For every $k \in \mathbb{N}$, put $a_k = b_1 \oplus \cdots \oplus b_k$. Evidently, a_k is a non-decreasing sequence, so $\lor (a_k)_{k \in \mathbb{N}}$ exists and equals $\bigoplus_{k \in \mathbb{N}} b_k$.

(a) \Rightarrow (b): Let $(a_k)_{k\in\mathbb{N}}$ be a non-increasing sequence. For every $k \in \mathbb{N}$, put $b_k = a_k \ominus a_{k+1}$; $(b_k)_{k\in\mathbb{N}}$ is the difference sequence of $(a_k)_{k\in\mathbb{N}}$. Evidently, $(b_k)_{k\in\mathbb{N}}$ is an orthogonal sequence. Denote $b = \bigoplus_{k\in\mathbb{N}} b_k$. We claim that $a_1 \ominus b = \bigwedge a_i$. Indeed, for all $k \in \mathbb{N}$ we have $a_k = a_1 \ominus (b_1 \oplus \cdots \oplus b_{k-1}) \ge a_1 \ominus b$. Thus, $a_1 \ominus b$ is a lower bound of $(a_k)_{k\in\mathbb{N}}$. On the other hand, let c be a lower bound of $(a_k)_{k\in\mathbb{N}}$. Then, for all $k \in \mathbb{N}$, $c \le a_k = a_1 \ominus (b_1 \oplus \cdots \oplus b_{k-1})$. This implies that $a_1 \ominus c \ge b_1 \oplus \cdots \oplus b_{k-1}$. Therefore, $a_1 \ominus c \ge b$ which is equivalent to $c \le a_1 \ominus b$.

(b) \Rightarrow (c): This is a consequence of the equivalence $a \leq b$ iff $b' \leq a'$.

Recall, that a lattice is called σ -complete iff every countable set of elements has a supremum.

Corollary 3.4. Let $(E; \oplus, 0, 1)$ be a lattice-ordered effect algebra. E is σ -complete as an effect algebra iff E is σ -complete as a lattice.

Proof. It is obvious that the σ -completeness of E as a lattice is equivalent to the condition (b) of Lemma 3.3.

Thus, the class of σ -complete effect algebras includes effect algebras arising from σ -complete MV-algebras and effect algebras arising from σ -complete orthomodular lattices.

Let *E* be a σ -complete effect algebra, let \sim be a weak congruence on *E*. Then \sim is said to be *countably additive* iff the following condition is satisfied: for all orthogonal families $(a_k)_{k\in\mathbb{N}} \subseteq E$ and $(b_k)_{k\in\mathbb{N}} \subseteq E$ such that, for all $k \in \mathbb{N}$, $a_k \sim b_k$ we have $\bigoplus_{k\in\mathbb{N}} a_k \sim \bigoplus_{k\in\mathbb{N}} b_k$.

Proposition 3. Let E be a σ -complete effect algebra. Let \sim be a countably additive weak congruence satisfying (C5). Then \leq is a partial order on E/\sim .

Proof. Let $a_0, b_0, c_0 \in P$ be such that $a_0 \sim a_0 \oplus b_0 \oplus c_0$. By Proposition 2, it suffices to prove that $a_0 \sim a_0 \oplus b_0$.

When we apply the (C5) property of ~ twice, we immediately obtain that $a_0 \sim a_0 \oplus b_0 \oplus c_0$ implies that there exist $a_1, b_1, c_1 \in P$ such that $a_1 \sim a_0, b_1 \sim b_0, c_1 \sim c$ and $a_0 = a_1 \oplus b_1 \oplus c_1$. Similarly, since $a_1 \sim a_1 \oplus b_1 \oplus c_1$, there exist $a_2, b_2, c_2 \in P$ such that $a_2 \sim a_1, b_2 \sim b_1$ and $c_2 \sim c_1$.

This way, we can construct sequences $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ such that

$$a_0 \oplus b_0 \ge a_0 \ge a_1 \oplus b_1 \ge a_1 \ge a_2 \oplus b_2 \ge \cdots \tag{3}$$

Moreover, for all $k \in \mathbb{N}$, $a_k \sim a_0$ and $b_k \sim b_0$. Since E is a σ -complete effect algebra, the non-decreasing sequence (3) possesses an infimum, say f. (See Lemma 3.3.)

Consider the following sequences $(d_k)_{k \in \mathbb{N}}, (e_k)_{k \in \mathbb{N}}$:

$d_0=(a_0\oplus b_0)\ominus a_0\sim b_0$	$e_0 = d_2$
$d_1 = a_0 \ominus (a_1 \oplus b_1) \sim c_0$	$e_1 = d_1$
$d_2=(a_1\oplus b_1)\ominus a_1\sim b_0$	$e_2 = d_4$
$d_3 = a_1 \ominus (a_2 \oplus b_2) \sim c_0$	$e_3 = d_3$
:	÷

Note that $(d_k)_{k \in \mathbb{N}}$ is the difference sequence of (3). Similarly, $(e_k)_{k \in \mathbb{N}}$ arises from the difference sequence of the sequence

$$a_0 \ge a_1 \oplus b_1 \ge a_1 \ge a_2 \oplus b_2 \ge \cdots \tag{4}$$

by swapping the elements with indices 2k and 2k + 1, for $k \in \mathbb{N}$.

Consider the proof of Lemma 3.3, part (a) \Rightarrow (b). We have

$$a_0 \oplus b_0 = \bigoplus_{k \in \mathbb{N}} d_k \oplus f$$
$$a_0 = \bigoplus_{k \in \mathbb{N}} e_k \oplus f$$

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Since for all $k \in \mathbb{N}$ $e_k \sim d_k$ and \sim is countably additive, this implies that $a_0 \oplus b_0 \sim a_0$.

Without going into details, it is worth mentioning that, in view of Example 3.2, Proposition 3 implies that for every Rickart *-ring in which every countable family of projections has a supremum, the set of equivalence classes of projections under the relation \sim is partially ordered. This was proved in [3] under the name Schröder-Bernstein theorem. (See also [17].)

4. Central elements; main result

Let E be an effect algebra. Suppose that there is an isomorphism $\phi: E \mapsto E_1 \times E_2$. For every such ϕ , the elements $\phi^{-1}(1,0)$ and $\phi^{-1}(0,1)$ are called *central elements* of E. We write C(E) for the set of all central elements of an effect algebra E. It was proved in [9] that the set of all central elements forms a subeffect algebra of E, which is a Boolean algebra. Moreover, the joins and meets of elements of C(E) exist in E and coincide with their joins and meets in C(E). If $a, b \in C(E)$ are orthogonal, we have $a \vee b = a \oplus b$ and $a \wedge b = 0$. For all $a \in C(E)$, the interval [0, a] is a \oplus -subalgebra and hence an ideal of E. These ideals are called *central ideals*. By [5], a central ideal in an effect algebra E can be characterized as an ideal I satisfying the following conditions.

- I = [0, a] for some $a \in E$.
- I is a *Riesz ideal*, i.e., if $i \in I$ and $i \leq a \oplus b$, then there exist $i_1, i_2 \in I$, such that $i_1 \leq a, i_2 \leq b, i \leq i_1 \oplus i_2$.

For every central element a, the map $x \mapsto a \wedge x$ is a full homomorphism, which maps E onto $[0, a]_E$ (cf. [14]).

Lemma 4.1. Let E be a σ -complete effect algebra. Let $(a_k)_{k\in\mathbb{N}} \subseteq E$ be a countable orthogonal family of central elements. Let $(x_k)_{k\in\mathbb{N}}$ be a family of elements satisfying $x_k \leq a_k$, for all $k \in \mathbb{N}$. Then $\bigvee_{k\in\mathbb{N}} x_k$ exists and equals $\bigoplus_{k\in\mathbb{N}} x_k$.

Proof. Let M be a finite nonempty subset of \mathbb{N} . Obviously, $(x_k)_{k \in M}$ is an orthogonal family, so $\bigoplus_{k \in M} x_k$ exists. Observe that

$$E \cong (\prod_{k \in M} [0, a_k]_E) \times [0, (a_1 \oplus \dots \oplus a_k)']_E.$$

Therefore, $\bigoplus_{k \in M} x_k = \bigvee_{k \in M} x_k$. This implies that

$$\oplus (x_k)_{k \in \mathbb{N}} = \bigvee_{k \in \mathbb{N}} x_1 \oplus \cdots \oplus x_k = \bigvee_{k \in \mathbb{N}} x_1 \vee \cdots \vee x_k = \bigvee_{k \in \mathbb{N}} x_k.$$

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Proposition 4. Let E be a σ -complete effect algebra. Let $(a_n)_{n \in \mathbb{N}} \subseteq C(E)$ be a countable orthogonal family. Denote $a = \bigoplus_{n \in \mathbb{N}} a_n$. Then

$$[0,a]_E \cong \prod_{n \in \mathbb{N}} [0,a_n]_E.$$

Moreover, a is central.

Proof. Let $\phi \colon [0, a]_E \mapsto \prod_{n \in \mathbb{N}} [0, a_n]_E$ be a map given by $\phi(x) = (x \wedge a_n)_{n \in \mathbb{N}}$. We will prove that ϕ is an isomorphism.

To prove that ϕ is onto, let $(x_k)_{k\in\mathbb{N}}\in\prod_{n\in\mathbb{N}}[0,a_n]_E$. Observe that (x_k) is an orthogonal family and put $x=\bigoplus_{n\in\mathbb{N}}x_n$. We will prove that $\phi(x)=(x_n)_{n\in\mathbb{N}}$. We see that

$$\phi(x) = (x \wedge a_n)_{n \in \mathbb{N}} = ((\bigoplus_{k \in \mathbb{N}} x_k) \wedge a_n)_{n \in \mathbb{N}}.$$

Fix any $n \in \mathbb{N}$. By associativity of \oplus ,

$$x \wedge a_n = (\bigoplus_{k \in \mathbb{N}} x_k) \wedge a_n = (x_n \oplus (\bigoplus_{k \in \mathbb{N} \setminus \{n\}} x_k)) \wedge a_n$$

Since the orthogonal family $(x_k)_{k \in \mathbb{N} \setminus \{n\}}$ satisfies the conditions of Lemma 4.1,

$$(x_n \oplus (\bigoplus_{k \in \mathbb{N} \setminus \{n\}} x_k)) \land a_n = (x_n \oplus (\bigvee_{k \in \mathbb{N} \setminus \{n\}} x_k)) \land a_n$$

Since a_n is a central element,

$$(x_n \oplus (\bigvee_{k \in \mathbb{N} \setminus \{n\}} x_k)) \wedge a_n = (x_n \wedge a_n) \oplus ((\bigvee_{k \in \mathbb{N} \setminus \{n\}} x_k) \wedge a_n)$$
$$= x_n \oplus ((\bigvee_{k \in \mathbb{N} \setminus \{n\}} x_k) \wedge a_n)$$

Since, for all $k \in \mathbb{N} \setminus \{n\}$, $x_k \wedge a_n = 0$, we see that

$$a_n \wedge (\bigvee_{k \in \mathbb{N} \setminus \{n\}} x_k) = 0.$$

Thus, for all $n \in \mathbb{N}$, $(x \wedge a_n) = x_n$. Therefore, ϕ is onto.

To see that ϕ is one-to-one it suffices to prove that, for all $x \in [0, a]$,

$$x = \bigoplus_{n \in \mathbb{N}} x \wedge a_n.$$
(5)

Since

$$\bigoplus_{n\in\mathbb{N}}x\wedge a_n=\bigvee_{n\in\mathbb{N}}(x\wedge a_1)\oplus\cdots\oplus(x\wedge a_n)=\bigvee_{n\in\mathbb{N}}x\wedge(a_1\oplus\cdots\oplus a_n),$$

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